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NOTE

HOARE'S LOGIC FOR PROGRAMMING LANGUAGES WITH TWO DATA TYPES*

J.A. BERGSTRA**
Department of Computer Science, Mathematical Centre, 1098 SJ Amsterdam, The Netherlands

J.V. TUCKER
Department of Computer Studies, University of Leeds, Leeds LS2 9JT, United Kingdom

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Abstract. We consider the completeness of Hoare's logic with a first-order assertion language applied to **while**-programs containing variables of two (or more) distinct types. Whilst Cook's completeness theorem generalizes to many-sorted interpretations, certain fundamentally important structures turn out not to be expressive. We study the case of programs with distinguished counter variables and Boolean variables adjoined; for example, we show that adding counters to arithmetic destroys expressiveness.

Key words. Hoare's logic, partial correctness, **while**-programs, completeness, expressiveness, many-sorted programs, many-sorted first-order logic.

Introduction

Since the publication of [6] there has accumulated a large body of knowledge about proof systems for formally verifying the partial correctness of programs. Proof systems have been made which include a wide variety of programming features and, in particular, the soundness and completeness of these systems have been successfully analysed along the lines first set down in [5]. To obtain information about what has been achieved, at least for the sequential control aspects of programming languages, see [1].

In this note we consider a simple feature of most programming languages which has gone unnoticed to date, namely the property that *there may be two (or more) distinct types of variable or identifier in a single program.* We demonstrate that whilst

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Cook's account of completeness generalizes to include Boolean variables, it is, surprisingly, unable to cope with **while**-programs with counters.

In Section 1 we summarize prerequisites and observe that Cook's completeness theorem for Hoare's logic for **while**-programs applied to first-order expressive structures generalizes to the many-sorted case. However, in Section 2, we prove that adding arithmetic N to an expressive structure A can lead to a non-expressive two-sorted interpretation $[A, N]$. In particular, we prove that adding arithmetic N to arithmetic N leads to a non-expressive structure $[N, N]$ and, indeed, that Hoare's logic for $[N, N]$ is incomplete (Theorem 2.3). Thus, there is a general completeness theorem for the two-type situation, but it cannot be applied to a canonical example.

1. Assertions, programs and Hoare's logic

In addition to necessary prerequisites about two-sorted syntax and semantics, we outline the fate of Cook's study [6] of Hoare's logic when generalized to the two-sorted situation as this is the background of our main results.

Syntax

The first-order language $L(\Sigma)$ of some two-sorted signature Σ is based upon two sets of variables

$$x_1^1, x_2^1, \ldots \quad \text{and} \quad x_1^2, x_2^2, \ldots,$$

of sorts 1 and 2 respectively, and the constant, function and relation symbols of $L(\Sigma)$ are those of Σ together with equality symbols of sorts 1 and 2.

The usual inductive definition of term now yields two kinds of term giving values of sort 1 and sort 2. Atomic formulae have the form

$$t^i =_i s^i \quad \text{and} \quad R(y_1^{i_1}, y_2^{i_2}, \ldots, y_k^{i_k})$$

where t^i, s^i are terms (having values) of sort i, $=_i$ is the equality symbol for sort i, R is a relation symbol and the $y_j^{i_j}$ are variables of sort i_j, $j = 1, \ldots, k$ and $i, i_j \in \{1, 2\}$.

The well-formed formulae of $L(\Sigma)$ are made inductively by applying the logical connectives $\wedge$, $\vee$, $\neg$, $\rightarrow$ and the quantifiers

$$\forall x_j^1 \quad \exists x_j^1 \quad \forall x_j^2 \quad \exists x_j^2 \qquad j \in \mathbb{N}$$

in the usual way.

Using the syntax of $L(\Sigma)$ the set $\mathrm{WP}(\Sigma)$ of all **while**-programs over Σ is defined in the obvious way. Note, in particular, that there are two kinds of assignment statement

$$x_j^1 := t^1 \quad \text{and} \quad x_j^2 := t^2$$

but that Boolean tests in control statements are simply quantifier-free formulae of $L(\Sigma)$ and may refer to both sorts.

By a *specified* or *asserted program* we mean a triple of the form $\{p\}S\{q\}$ where $p, q \in L(\Sigma)$ and $S \in \mathrm{WP}(\Sigma)$.

Semantics

The semantics of $L(\Sigma)$ is based on two-sorted structures A of signature Σ and is formally defined in the usual manner.

The set of all sentences of $L(\Sigma)$ which are true in structure A is called the first-order theory of A and is denoted $\mathrm{Th}(A)$. For $\phi \in L(\Sigma)$ the set defined in A by ϕ we denote $\phi[A]$.

For the semantics of $\mathrm{WP}(\Sigma)$ on an interpretation A we leave the reader free to choose any sensible account of **while**-program computation in one-sorted structures and then to generalize it. Certainly, the operational and denotational semantics given in [2] have natural many-sorted generalizations (see [8]).

We suppose that the meaning of $S \in \mathrm{WP}(\Sigma)$ on interpretation A is defined as a state transformation

$$M_A(S) : \text{STATES}(A) \to \text{STATES}(A).$$

Also if S has n variables of sort 1 and m variables of sort 2, then $\text{STATES}(A) \cong A_1^n \times A_2^m$, where A_1, A_2 are the domains of sorts 1, 2 in A, and we suppose that $M_A(S)$ is represented by a mapping

$$\hat{M}_A(S) : A_1^n \times A_2^m \to A_1^n \times A_2^m.$$

Putting together the semantics of $L(\Sigma)$ and $\mathrm{WP}(\Sigma)$ we consider the partial correctness semantics of the specified programs: $\{p\}S\{q\}$ is valid on A, written $A \models \{p\}S\{q\}$, if when p is true, then either S diverges or S converges to a state at which q is true. The set of all specified programs valid on A is called the partial correctness theory of A and we write

$$\mathrm{PC}(A) = \{\{p\}S\{q\} : A \models \{p\}S\{q\}\}.$$

Hoare's logic

Hoare's logic for the two-sorted $\mathrm{WP}(\Sigma)$ has exactly the same axiom scheme for assignment statements and the same rules for composition, conditionals and iteration. In addition, any first-order theory T may be employed to prove a specification for the underlying data types and T affects program correctness proofs via the Rule of Consequence (see [5, 6]). The set of all specified programs provable from T is denoted $\mathrm{HL}(T)$.

In this note we are interested in proving correctness with respect to a given two-sorted structure A. Cook's work on the single-sorted version of this case generalizes to provide us with the following account.

1.1. Soundness Theorem. *If* $A \models T$, *then* $\mathrm{HL}(T) \subset \mathrm{PC}(A)$.

The assertion language $L(\Sigma)$ is said to be expressive for $\mathrm{WP}(\Sigma)$ over A if for any $p \in L(\Sigma)$ and $S \in \mathrm{WP}(\Sigma)$ there is a formula $\mathrm{SP}(p, S) \in L(\Sigma)$ that defines the strongest postcondition $\mathrm{SP}_A(p, S)$ of S with respect to p over A,

$$\mathrm{SP}_A(p, S) = \{\sigma \in \mathrm{STATES}(A): \exists\tau\, [M_A(S)(\tau){\downarrow}\sigma \,\&\, p(\tau)]\}.$$

Notice that expressiveness is actually a property of the interpretation A rather than $L(\Sigma)$. We call HL complete for A if $\mathrm{HL}(\mathrm{Th}(A)) = \mathrm{PC}(A)$.

1.2. Cook's Completeness Theorem. *Suppose* $L(\Sigma)$ *is expressive for* $\mathrm{WP}(\Sigma)$ *over* A *and let* $T = \mathrm{Th}(A)$. *Then* $\mathrm{HL}(T) = \mathrm{PC}(A)$.

In view of Theorem 1.2 we define $\mathrm{HL}(A) = \mathrm{HL}(\mathrm{Th}(A))$, and observe that $\mathrm{HL}(A)$ represents the strongest Hoare logic for analyzing correctness on A because it is equipped with all first-order true facts about A.

1.3. Theorem. *If* A *is finite, then* A *is expressive and* $\mathrm{HL}(A)$ *is complete.*

2. Adding arithmetic

Semantically, adding counters to **while**-programs is effected by interpreting the two-sorted programming language $\mathrm{WP}(\Sigma)$ on certain two-sorted structures of the following form.

Let A and B be single-sorted structures with disjoint signatures Σ_A and Σ_B respectively. Then we define the *join* $[A, B]$ of A and B to be the two-sorted structure of signature $\Sigma_{A,B} = \Sigma_A \cup \Sigma_B$ whose disjoint domains and operations are simply those of A and B.

What is noteworthy in this operation on structures is that algebraically A and B remain independent data types. Adding arithmetic means computing on structures $[A, \mathbb{N}]$ where $\mathbb{N}$ is the standard model of arithmetic. Adding Booleans means computing on structures $[A, \mathbb{B}]$ where $\mathbb{B} = \{\mathrm{tt}, \mathrm{ff}\}$ equipped with $\wedge$, $\neg$.

We prove that Hoare's logic is incomplete when applied to structures $[A, \mathbb{N}]$.

2.1. Proposition. *If* $[A, B]$ *is expressive, then* A *and* B *are expressive.*

Proof. We begin by stating a basic fact about first-order definability on $[A, B]$.

Let H be the smallest set of $\Sigma_{A,B} = \Sigma_A \cup \Sigma_B$ formulae that contains $L(\Sigma_A)$ and $L(\Sigma_B)$ and is closed under $\neg$, $\wedge$, $\vee$. Thus, H does *not* contain formulae with quantifiers ranging over different sorts such as

$$\forall x^A\, (\phi^A \wedge \phi^B).$$

2.2. Separation of Variables Lemma. *Each formula* $\phi \in L(\Sigma_{A,B})$ *is equivalent to a formula of* H.

Proof. The proof follows by induction on the structure of ϕ (see [3]). □

Proof of Proposition 2.1 (*continued*). To prove the proposition we assume $[A, B]$ is expressive and prove that A is expressive (the case for B follows mutatis nomine).

Let $\phi \in L(\Sigma_A)$ and $S \in \mathrm{WP}(\Sigma_A)$. Let $\mathrm{SP}(\phi, S)$ define the strongest postcondition $\mathrm{SP}_{[A,B]}(\phi, S)$ on $[A, B]$. By the Separation of Variables Lemma 2.2,

$$\mathrm{SP}(\phi, S) \equiv \bigvee_{i=1}^{s} (\psi_i^A \wedge \psi_i^B).$$

Because ϕ and S involve variables of type A only, the components ψ_i^B for $1 \leq i \leq s$ are closed and can be replaced by their propositional values **true** and **false**. This being done we obtain a formula $\psi \in L(\Sigma_{A,B})$, equivalent to $\mathrm{SP}(\phi, S)$, that is first-order over Σ_A and, indeed, ψ defines $\mathrm{SP}_A(\phi, S)$ on A. □

Our main result implies that the converse of Proposition 2.1 is false. Let $\mathbb{N}$ denote standard model of arithmetic; to be precise let

$$\mathbb{N} = (\{0, 1, \ldots\}, 0, 1, x+1, x \dot{-} 1, x+y, x \cdot y).$$

Consider the structure $[N_1, N_2]$ of signature $\Sigma_{1,2}$ wherein $N_1 = \mathbb{N}$ has signature Σ_1 and $N_2 = \mathbb{N}$ has signature Σ_2, i.e., $[N_1, N_2]$ is a pair of algebraically independent copies of $\mathbb{N}$. We are looking at the case of adding arithmetic to arithmetic, so to say.

2.3. Theorem. *The two-sorted structure* $[N_1, N_2]$ *is not expressive and* $\mathrm{HL}([N_1, N_2])$ *is not complete.*

Proof. Consider the following program:

$S ::= \quad x := 0;\ z := 0;$

$\quad\quad$ **while** $x \neq y$ **do** $x := x+1;\ z := z+1$ **od**

with x, y variables of sort 1 and z a variable of sort 2. The strongest post-condition of S with respect to **true** is

$$\mathrm{SP}(\mathbf{true}, S) = \{(a, b, c) \in N_1 \times N_1 \times N_2 \colon a = b = c = n \in \mathbb{N}\}.$$

Suppose $\mathrm{SP}(\mathbf{true}, S)$ is first-order definable over $[N_1, N_2]$; then clearly the 'diagonal' $\Delta = \{(a, b) \in N_1 \times N_2 \colon a = b = n \in \mathbb{N}\}$ is first-order definable: to this latter statement we derive a contradiction.

By the Separation of Variables Lemma 2.2, it is sufficient to show that Δ is not definable by a formula of $H(\Sigma_{1,2})$.

Suppose as a contradiction that Δ is definable by $\phi \in H(\Sigma_{1,2})$ with free variables x, y of sorts 1, 2; thus,

$$\Delta = \{(a, b) \in N_1 \times N_2 \colon [N_1, N_2] \models \phi(a, b)\}.$$

Now ϕ can be written in disjunctive normal form:

$$\phi \equiv \bigvee_{i=1}^{s} \bigwedge_{j=1}^{t} \phi_{i,j},$$

where $\phi_{i,j} \in L(\Sigma_1) \cup L(\Sigma_2)$ for $1 \leq i \leq s$ and $1 \leq j \leq t$. This can be compressed to

$$\phi \equiv \bigvee_{i=1}^{s} (\Phi_i^1 \wedge \Phi_i^2),$$

where $\Phi_i^1 \in L(\Sigma_1)$ and $\Phi_i^2 \in L(\Sigma_2)$ with free variables x and y respectively. For $1 \leq i \leq s$, set

$$\Delta_i = \{(a, b) \in N_1 \times N_2 \colon [N_1, N_2] \models \Phi_i^1(a) \wedge \Phi_i^2(b)\},$$

so that $\Delta = \bigcup_{i=1}^{s} \Delta_i$. At least one Δ_i is infinite, say Δ_0. We choose two points (a, a), $(b, b) \in \Delta_1$ with $a \neq b$. Now

$$[N_1, N_2] \models \Phi_0^1(a) \wedge \Phi_0^2(a) \quad \text{and} \quad [N_1, N_2] \models \Phi_0^1(b) \wedge \Phi_0^2(b).$$

Thus,

$$[N_1, N_2] \models \Phi_0^1(a) \wedge \Phi_0^2(b).$$

This means that $(a, b) \in \Delta_0 \subset \Delta$ which is not the case. Therefore, $[N_1, N_2]$ is not expressive.

In order to see that HL($[N_1, N_2]$) is not complete, consider the program

$$S_2 ::= \textbf{while } x \neq 0 \wedge y \neq 0 \wedge z \neq 0$$
$$\textbf{do } x := x \dot{-} 1;\ y := y \dot{-} 1;\ z := z \dot{-} 1 \textbf{ od}.$$

Clearly,

$$[N_1, N_2] \models \{\textbf{true}\}\ S_1\ ;\ S_2\ \{x = 0 \wedge y = 0 \wedge z = 0\}.$$

In order to prove this valid asserted program using Hoare's logic, an intermediate assertion θ must be found, i.e., a formula such that

$$[N_1, N_2] \models \{\textbf{true}\}\ S_1 \{\theta\}, \qquad [N_1, N_2] \models \{\theta\}\ S_2 \{x = 0 \wedge y = 0 \wedge z = 0\}.$$

Thus,

$$\text{SP}_{[N_1, N_2]}(\textbf{true}, S_1) \subset \theta[N_1, N_2] \subset \text{WP}_{[N_1, N_2]}(S_2, x = 0 \wedge y = 0 \wedge z = 0).$$

But

$$\text{WP}_{[N_1, N_2]}(S_2, x = 0 \wedge y = 0 \wedge z = 0) = \text{SP}_{[N_1, N_2]}(\textbf{true}, S_1)$$

and hence $\theta[N_1, N_2] = \text{SP}(\textbf{true}, S_1)$. This contradicts the fact that SP(**true**, S_1) is not definable. □

3. Concluding remarks

Quite clearly no useful account of the correctness of many-typed programs can be founded on a first-order assertion language. Fortunately, it is possible to give a very thorough theory of the partial and total correctness of the basic sequential constructs in a many-sorted abstract setting if one allows the extension to a weak second-order assertion language (see [8]). Moreover, allowing hidden functions to enhance expressiveness is certainly an acceptable step; for initial algebra specification it is required.

In contrast to Theorem 2.3 one can show the following theorem.

Theorem. *If A is expressive and F is finite, then $[A, F]$ is expressive and consequently* HL($[A, F]$) *is complete.*

Finally it should be pointed out that, in logic, preservation theorems are known for products (cf. the theorem of Feferman and Vaught as in [7]); such properties still have to be established for program verification logics.

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